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Optimality of Proportional Navigation

ELIEZER KREINDLER*

Grumman Aerospace Corporation, Bethpage, N.Y.

1. Introduction and Summary

PROPORTIONAL navigation is a well-known homing intercept guidance law, where the rate of turn of the interceptor (e.g., a missile) is made proportional to the rate of turn of the line of sight (LOS) between the interceptor and the target (e.g., an airplane).[11] The differential equations of motion governing the intercept geometry (see Sec. 2) can be linearized by assuming that the angle $\sigma(t)$ of the rotation of LOS is small and the closing velocity v_c along the LOS is constant. Thus, several authors were led to explore the application of linear optimal control to the homing guidance problem,[1–10] using mostly the performance index

$$I = cx_1^2(t_1)+\int_0^{t_1} u^2(t)\,dt \tag{1}$$

which balances, by a positive weighting factor c, the terminal miss $x_1(t_1)$ with the integral square of the interceptor's normal commanded acceleration $u(t)$.

When target maneuvers and interceptor dynamics are neglected, minimization of Eq. (1) results in the guidance law[1–3]

$$u(t) = -\frac{3v_c(t_1-t)^3}{3/c+(t_1-t)^3}\,\dot\sigma(t) \tag{2}$$

For $c\to\infty$ the dependence on the time-to-go, t_1-t, cancels out,

$$\lim_{c\to\infty} u(t) = \begin{cases} -3v_c\dot\sigma(t) & 0\leqq t<t_1 \\ 0 & t=t_1 \end{cases} \tag{3}$$

resulting in a proportional navigation law[11] with navigation constant $N=3$ [except at $t=t_1$; however, for $N=3$, $\dot\sigma(t)\to 0$ as $t\to t_1$ so that $u(t)$ is continuous at $t=t_1$].

The result, Eq. (3), a constant optimal guidance law on a finite time-interval, is quite remarkable, and it invites the question: Are there indices in the general class

Received October 26, 1972; revision received January 22, 1973.

Index category: Launch Vehicle and Missile Guidance Systems.

* Research Scientist; now with the Faculty of Electrical Engineering, Israel Institute of Technology, Haifa, Israel.

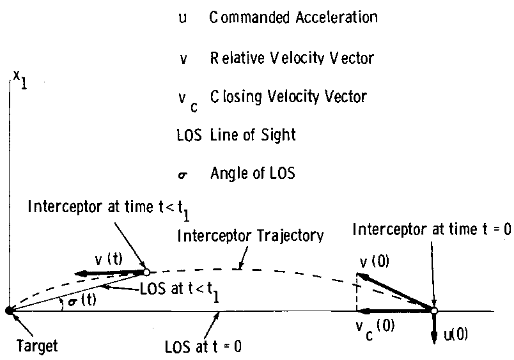

Fig. 1 Intercept geometry.

$$I = \tfrac{1}{2}x^T(t_1)Fx(t_1)+\tfrac{1}{2}\int_0^{t_1}[x^T(t)Q(t)x(t)+r(t)u^2(t)]\,dt \tag{4}$$

that result in the proportional navigation law

$$u(t) = -Nv_c\dot\sigma(t) \quad 0\leqq t\leqq t_1 \tag{5}$$

for N other than $N=3$? In Eq. (4), $x(t)$ is the state of the equations of relative motion; $u(t)$ is the scalar control variable, the commanded acceleration; F and $Q(t)$ are symmetric matrices and $r(t)$ is positive; superscript T denotes the matrix transpose.

This is an inverse problem in optimal control, more precisely defined in Sec. 2. The general class of indices (4) is constructed in Sec. 3, and one special case, possibly the simplest, is given by

$$F=0,\ Q(t)=\begin{bmatrix} N(N+2) & 0 \\ 0 & N(N+1)(t_1-t)^2 \end{bmatrix},\quad r(t)=(t_1-t)^4 \tag{6}$$

The potential practical significance of this result is discussed in Sec. 4. The case where $r(t)$ is constant represents a generalization of the index (1); the resulting performance index is recorded in the Appendix.

2. Formulation

Consider a target moving with constant velocity. If the interceptor's commanded acceleration vector lies (as it should and as is assumed to lie) in the plane of relative motion defined by the initial line of sight (LOS) and the interceptor's initial relative velocity vector $v(0)$, then the relative motion remains in this plane. This is depicted in Fig. 1, which lies in the plane of relative motion and where the coordinate system moves with the target's constant velocity (which may be assumed zero).

With $u(t)\equiv 0$, intercept is possible if, and only if, $v(t)$ lies along the LOS, and then $\dot\sigma(t)\equiv 0$. In proportional navigation, $u(t)$ is commanded according to (1.5) with $N>2$, and then $\dot\sigma(t)$ is nulled at the intercept time $t=t_1$.

Assuming $\sigma(t)$ is small, the commanded acceleration $u(t)$ and the closing velocity $v_c(t)$ (the component of $v(t)$ along the LOS) can be approximated as being, respectively, normal to and along the initial LOS. Thus $v_c(t)$ is constant and only the motion relative to the x_1-axis need be considered

$$\dot x_1(t) = x_2(t),\quad \dot x_2(t) = u(t) \tag{7}$$

Further, the LOS angle $\sigma(t)$ can be approximated by

$$\sigma(t)\cong x_1(t)/(t_1-t)v_c \tag{8}$$

whence the proportional navigation guidance law (5) becomes

$$u(t) = -\frac{N}{(t_1-t)^2}x_1(t)-\frac{N}{(t_1-t)}x_2(t),\quad 0\leqq t\leqq t_1 \tag{9}$$

We write the equations of motion (7) as

$$\dot x(t) = Ax(t)+bu(t) \tag{10}$$

and the proportional navigation law (9) as

$$u(t) = -k^T(t)x(t) \tag{11}$$

where

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} b = \begin{bmatrix} 0 \\ 1 \end{bmatrix} k = \begin{bmatrix} N/(t_1-t)^2 \\ N/(t_1-t) \end{bmatrix} \tag{12}$$

The inverse problem is to find performance indices of the form (4) for which the guidance law (11) is optimal.

3. Derivation of Performance Indices

For the control law (11) to be optimal, $k(t)$ must satisfy[3]

$$k(t) = r^{-1}(t)P(t)b \tag{13}$$

$$-\dot{P}(t) = P(t)A + A^T P(t) - r(t)k(t)k^T(t) + Q(t), \quad P(t_1) = F \tag{14}$$

The general symmetric $P(t)$ that satisfies Eq. (13) is by inspection

$$P(t) = \begin{bmatrix} p_{11}(t) & r(t)N/(t_1-t)^2 \\ r(t)N/(t_1-t)^2 & r(t)N/(t_1-t) \end{bmatrix} \tag{15}$$

where $p_{11}(t)$ is arbitrary. Substituting $P(t)$ and $\dot{P}(t)$ in Eq. (14) yields

$$Q(t) = \begin{bmatrix} -\dot{p}_{11}(t) + \dfrac{r(t)N^2}{(t_1-t)^4} & -p_{11}(t) + \dfrac{r(t)N(N-2)}{(t_1-t)^3} - \dfrac{\dot{r}(t)N}{(t_1-t)^2} \\ -p_{11}(t) + \dfrac{r(t)N(N-2)}{(t_1-t)^3} - \dfrac{\dot{r}(t)N}{(t_1-t)^2} & \dfrac{r(t)N(N-3)}{(t_1-t)^2} - \dfrac{\dot{r}(t)N}{(t_1-t)} \end{bmatrix} \tag{16}$$

To verify that the index (4) is actually minimized when $P(t)$, $Q(t)$, and $r(t)$ satisfy Eqs. (13 and 14), we multiply Eq. (14) on both sides by x, and use $Ax(t) = \dot{x}(t) - bu(t)$ to obtain

$$-\frac{d}{dt}[x^T(t)P(t)x(t)] = -r(t)[u(t)+k^T(t)x(t)]^2 + x^T(t)Q(t)x(t) + r(t)u^2(t) \tag{17}$$

whence by integration and setting $P(t_1) = F$

$$\tfrac{1}{2}x^T(0)P(0)x(0) + \tfrac{1}{2}\int_0^{t_1} r(t)[u(t)+k^T(t)x(t)]^2\, dt = \tfrac{1}{2}x^T(t_1)Fx(t_1) + \tfrac{1}{2}\int_0^{t_1} [x^T(t)Q(t)x(t) + r(t)u^2(t)]\, dt \tag{18}$$

Since $r(t) > 0$, the left side of Eq. (18)—and hence the index (4) which is the right side of Eq. (18)—is minimized if and only if $u(t) = -k^T(t)x(t)$.

This completes the solution of the inverse problem. Many different $[F, Q(t), r(t)]$ can be obtained from Eqs. (15 and 16) by a selection of $p_{11}(t)$ and $r(t)$; however, only the case of a diagonal $Q(t)$ will be considered here.

If $r(t)$ is constant, the stipulation of a diagonal $Q(t)$ determines $p_{11}(t)$; the resulting index is given in the Appendix. Evidently, to avoid limits and infinite elements in F and $Q(t)$, the weighting factor $r(t)$ should be time varying. Consider the simplest form

$$r(t) = (t_1-t)^m \tag{19}$$

Then, for

$$p_{11}(t) = N(1+(m-2)N)(t_1-t)^{m-3} \tag{20}$$

we have

$$q_{12}(t) = q_{21}(t) = 0 \tag{21}$$

$$q_{11}(t) = [(m-2)N^2 + (m^2-5m+6)N](t_1-t)^{m-4} \tag{22}$$

$$q_{22}(t) = [N^2 + (m-3)N](t_1-t)^{m-2} \tag{23}$$

which for $m = 4$ yields the index (6). From the solutions $x(t)$ and $u(t)$ of (10) and (11), given by (A1–A3) in the Appendix, it is seen that the index (6) is finite for any $N \geqq 0$; for $u(t)$ to remain finite, $N \geqq 2$ is required, as is well-known[11] and is evident in (A3).

4. Discussion

The performance index defined by Eq. (6) is preferable to the index (1), because it produces the proportional navigation law (6) for any positive N, not just $N = 3$. On the other hand, the index (1) is easier to interpret physically.

Another preferable alternative to the performance index (1) is to let $c = 0$ and impose the constraint $x_1(t_1) = 0$;[3] then, Eq. (9) with $N = 3$ minimizes Eq. (1) with $c = 0$.[10] This can be generalized for $N \neq 3$ by the inverse process, but, save for the possibility of a constant $r(t)$, it does not appear to lead to an index simpler than (6).

The performance index (1) has been employed in more realistic settings that include interceptor dynamics (Refs. 7–9), and impressive reductions of target miss over the use of proportional navigation have been reported in Refs. 7 and 9 (not in Ref. 8). Of course, the price is a more complex guidance law that depends on the time-to-go, a quantity that must be estimated. It is of interest similarly to explore indices of the type (6).

Appendix: Generalization of the Index (1)

The solution $x(t)$ and $u(t)$ of Eqs. (10–12) is given by

$$x_1(t) = \left[\frac{N(t_1-t)}{(N-1)t_1} - \frac{(t_1-t)^N}{(N-1)t_1{}^N}\right]x_1(0) + \left[\frac{(t_1-t)}{(N-1)} - \frac{(t_1-t)^N}{(N-1)t_1{}^{N-1}}\right]x_2(0) \tag{A1}$$

$$x_2(t) = \left[\frac{N(t_1-t)^{N-1}}{(N-1)t_1{}^N} - \frac{N}{(N-1)t_1}\right]x_1(0) + \left[\frac{N(t_1-t)^{N-1}}{(N-1)t_1{}^{N-1}} - \frac{1}{N-1}\right]x_2(0) \tag{A2}$$

$$u(t) = -Nt_1{}^{-N}(t_1-t)^{N-2}[x_1(0) + t_1 x_2(0)] \tag{A3}$$

When $r(t) = 1$ and $p_{11}(t)$ is such that $Q(t)$ is diagonal, then Eq. (15) and Eq. (21) become

$$P(t) = \begin{bmatrix} \dfrac{N(N-2)}{(t_1-t)^3} & \dfrac{N}{(t_1-t)^2} \\ \dfrac{N}{(t_1-t)^2} & \dfrac{N}{(t_1-t)} \end{bmatrix}, \quad Q = \begin{bmatrix} \dfrac{-2N(N-3)}{(t_1-t)^4} & 0 \\ 0 & \dfrac{N(N-3)}{(t_1-t)^2} \end{bmatrix} \tag{A4}$$

For $N = 3$, Eq. (A4) reduces to the index (1) but with $F = \infty$ rather than

$$F = \begin{bmatrix} c & 0 \\ 0 & 0 \end{bmatrix} \tag{A5}$$

This is not surprising because, as noted in Eq. (3), the index (1) with $c = \infty$ does not produce the control law (9) at $t = t_1$. To produce a valid generalization of the index (1), we consider a finite c.

For finite c, the index (1) is minimized by

$$u(t) = -\frac{N(t_1-t)}{N/c+(t_1-t)^3}x_1(t) - \frac{N(t_1-t)^2}{N/c+(t_1-t)^3}x_2(t) \tag{A6}$$

with $N = 3^{1-3}$. Starting the inverse construction (as in Sec. 3) with this guidance law, and setting

$$p_{11}(t) = \frac{N[N/c+(t_1-t)^3] + N(N-3)(t_1-t)^3}{[N/c+(t_1-t)^3]^2} \tag{A7}$$

so that Q is diagonal, we obtain $P(t_1) = F$ given by Eq. (A5) and Q given by

$$q_{12}(t) = q_{21}(t) = 0 \tag{A8}$$

$$q_{11}(t) = \frac{4N(N-3)[N/c+(t_1-t)^3](t_1-t)^2 - 6N(N-3)(t_1-t)^5}{[N/c+(t_1-t)^3]^3} \tag{A9}$$

$$q_{22}(t) = \frac{N(N-3)(t_1-t)^4}{[N/c+(t_1-t)^3]^2} \tag{A10}$$

For $N = 3$, $Q = 0$, and we revert to the index (1). For $N \neq 3$ and $c = \infty$, this $Q(t)$ reduces to that in (A4), except for $t = t_1$ where $Q(t_1) \to 0$, and Eq. (A6) becomes

$$u(t)=\begin{cases}-Nv_c\dot{\sigma}(t), & 0\leqq t<t_1\\ 0, & t=t_1\end{cases} \tag{A11}$$

Thus, the index (4) with $r(t)=1$, F given by Eq. (A5), and Q given by Eqs. (A8–A10) is a generalization of the index (1), in the sense that when $N=3$ it reduces to Eq. (1) and when $c\to\infty$ it leads to the guidance law (A11), which is a generalization of Eq. (3) for any $N>0$.

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Local Potential Variational Method Applied to Hiemenz Flow

ROBERT T. DOTY* AND EDWARD F. BLICK†
University of Oklahoma, Norman, Okla.

Introduction

STAGNATION in plane flow, so-called "Hiemenz flow," has a well-known numerical solution due to Howarth[1] who improved the original solution by Hiemenz.[2] This problem offers an excellent opportunity to demonstrate the success that one of the more recent variational methods has had in providing simple approximate analytical expressions that are often very accurate when compared with "exact" numerical results. The method of analysis utilizes the concept of the local potential which is due to Glansdorff and Prigogine.[3] This method is sometimes called the "generalized entropy method" since it evolved from the more restrictive "principle of minimum entropy production." This development is summarized by Schechter,[4] and the analysis that follows is an extension of his work.

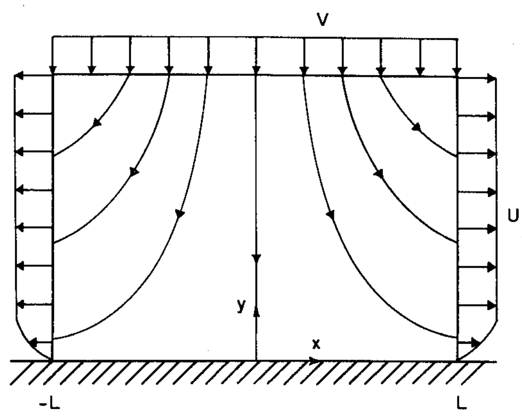

Fig. 1 Hiemenz flow pattern.

Variational Analysis

Consider the semi-infinite region near a stagnation point in plane flow bounded by $-L\leqq x\leqq L$, $0\leqq y\leqq\infty$ as shown in Fig. 1, where x is the coordinate measured parallel to the wall and y is the coordinate measured perpendicular to the wall. Then the equation for the local potential, E^*, as given by Schechter[4] for boundary-layer flow is

$$E^*=\iint\left[-\rho u^0u^0\frac{\partial u}{\partial x}-\rho u^0v^0\frac{\partial u}{\partial y}+\frac{\mu}{2}\left(\frac{\partial u}{\partial y}\right)^2-\rho\left(\frac{u^2}{2}+\frac{v^2}{2}\right)\left(\frac{\partial u^0}{\partial x}+\frac{\partial v^0}{\partial y}\right)+u\frac{\partial p}{\partial x}\right]dy\,dx+\oint\left[\rho u^0u^0u\,dy+\rho u^0v^0u\,dx-\mu\frac{\partial u^0}{\partial y}u\,dx\right] \tag{1}$$

where the area integration is carried out over the semi-infinite region described above and the line integral is evaluated around the perimeter of this region. Here u and v are the components of the viscous fluid velocity in the x and y directions, respectively, and ρ, μ, and p, are the density, viscosity and pressure of the fluid as usual. The superscript zero denotes a stationary state value. Continuity and the usual no-slip and infinity conditions reduce Eq. (1) to

$$E^*=\int_{-L}^{L}\int_0^{\infty}\left[-\rho u^0u^0\frac{\partial u}{\partial x}-\rho u^0v^0\frac{\partial u}{\partial y}+\frac{\mu}{2}\left(\frac{\partial u}{\partial y}\right)^2+u\frac{\partial p}{\partial x}\right]\times dy\,dx+2\int_0^{\infty}\rho u^0u^0u|_{x=L}\,dy \tag{2}$$

We now make the viscous flow assumptions for plane flow near a stagnation point as given by Schlichting.[5] Namely, we assume that

$$u=xf'(y),\quad v=-f(y),\quad \partial p/\partial x=-\rho a^2x \tag{3}$$

where $f(y)$ is to be determined, and a is a constant which is given by the inviscid potential flow near the stagnation point. If we let U and V be the inviscid velocity components, then near the plane stagnation point the potential flow is

$$U=ax,\quad V=-ay \tag{4}$$

Since u must approach U when y is large, and is expected to exhibit an exponential decay when y is small, a reasonable choice for $f'(y)$ is given by

$$f'(y)=a(1-e^{-by}) \tag{5}$$

where the parameter b now describes the viscous flow and is to be determined by the variational method. Satisfying continuity then gives the following expressions for the viscous velocity components:

$$u=ax(1-e^{-by}),\quad v=-a[y-(1-e^{-by})/b] \tag{6}$$

Received October 16, 1972; revision received January 15, 1973.

Index category: Boundary Layers and Convective Heat Transfer—Laminar.

* Graduate Student, Department of Aerospace, Mechanical and Nuclear Engineering.

† Professor, Department of Aerospace, Mechanical and Nuclear Engineering. Member AIAA.